Does the Schrödinger wave function describe a real physical wave?

Kurt Jung

Fachbereich Physik, Technische Universität Kaiserslautern,
Erwin-Schrödinger-Str. 56, D-67663 Kaiserslautern, Germany,
E-mail: jung@physik.uni-kl.de

Abstract. In 1924 Louis de Broglie presented the phase wave model [1]. He postulated that particle and wave coexist. Both objects are coupled to each other by the phase harmony principle, which always tends to synchronize the oscillations of wave and particle.

The escort wave model [2] replenishes the phase wave model by attributing a stochastic velocity component and a potential dependent rest mass to the particle. Moreover it introduces a novel method to deal with standing waves. In this new model a particle's trajectory is continuous but not deterministic because the particle is not only subjected to regular forces derived from external potentials but also to fluctuating forces. On this condition the space dependence of the wave escorting a particle agrees with the Schrödinger wave function.

By a measurement process the phase harmony coupling is broken off. Contrary to the standard interpretation of quantum mechanics the escort wave does not collapse but disappears in the ocean of fluctuating waves, because it is no longer stabilized by the particle. The particle immediately starts to build up a new escort wave by adjusting its phase to the phases of fluctuating waves, whose frequencies and rest systems nearly agree with the corresponding magnitudes of the particle.

1. Introduction

Since modern quantum mechanics has been established by Werner Heisenberg and Erwin Schrödinger numerous suggestions have been made how the abstract terms in the Hilbert space formalism have to be interpreted. Until now no satisfactory answer has been presented. Actually we do not need a new interpretation but we have to unify quantum mechanics with classical physics. If this is possible the interpretation will become evident.

Two approaches are carried out at the moment. Several physicists, e.g. [3] [4], study decoherence effects caused by the interaction of microscopic systems with the environment in order to explain why interference effects are not observed in macroscopic physics. These scientists assume that quantum mechanics is the universal theory and classical physics is a special case only valid for macrosystems.

The escort wave model [2] is based on the idea that classical physics is the groundwork of physics and that quantum phenomena arise from the interplay of particles with their associated waves. This statement cannot be proved within the quantum mechanical formalism but one has to start from conceptions used in classical physics. The uncertainty relation implies that position and momentum of a particle cannot be exactly determined at the same time. This does not necessarily means that both magnitudes cannot be always well defined. In the escort wave model it is taken for granted that particles always move on continuous trajectories. In order to

Journal of Physics: Conference Series **306** (2011) 012071 doi:10.1088/1742-6596/306/1/012071

account for the fact that quantum mechanics contains stochastic elements it will be assumed that particles are subjected to nonclassical forces exerted by fluctuating waves.

Many scientists have introduced stochastic elements into quantum mechanics. Most of these attempts have been performed in combination with the guidance wave concept. Especially Bohm [5], de Broglie [6, 7] and Vigier [8–11] have to be mentioned here. Vigier was able to reproduce quantum mechanical results on the assumption that massive particles run on continuous trajectories. Because Vigier only looked for diffraction experiments, he did not realize that continuous trajectories are in conflict with Born's probability density interpretation of excited states.

The influence of fluctuating waves will be the stronger the better energy and momentum of particle and fluctuating wave agree. Because of the limited coherence length of the waves, which will be discussed below, the exerted forces will smoothly vary in time causing unpredictable modifications of the trajectory's curvature. Thus the non-deterministic trajectories remain continuously differentiable. Even if a single particle is studied the physical system is not isolated from the rest of the universe.

2. The escort wave associated with a free particle

For free particles the escort wave model is equivalent to the phase wave model presented by Louis de Broglie 1924 in his doctor thesis [1]. Louis de Broglie ascribes to each particle with finite rest mass a periodic phenomenon, which will be called in the following internal oscillation. The angular frequency ω of the internal oscillation is attained by combining Planck's law $E = \hbar\omega$ and Einstein's relation $E = mc^2$ with the total energy E, Planck's constant $\hbar = h/(2\pi)$ and the velocity of light c. A resting particle with mass m_0 and eigen frequency

$$\omega_0 = m_0 c^2/\hbar \tag{1}$$

is always escorted by a wave of frequency ω_0 and infinite wave length. For a particle at rest the escort wave has the form

$$\psi(x,t) = \phi e^{-i\omega_0 t}. \tag{2}$$

The minus sign in the exponent has been chosen in order to obtain a positive energy, if the energy operator $i\hbar\partial/\partial t$ is applied to the wave. The plus sign in the exponent is then associated with the corresponding antiparticle. Particle and wave are coupled to each other by the phase harmony principle, which always tends to adjust the particle's internal oscillation to the oscillation of the escorting wave at the location of the particle.

In the near zone of scattering and diffraction experiments the oscillations of particle and wave are most often out of phase. Thus trajectory and wave will be steadily modified in order to adjust the phases. Although the total deflection of the trajectory due to the particle wave interaction will generally be small the diffraction pattern may be considerably modified.

Figure 1, which summarizes the first two diagrams of de Broglie's thesis, shows the trajectory of a freely moving particle and three phase lines of the escorting wave. The trajectory is subdivided by circles in the distance of the internal oscillation period $c\tau_0 = 2\pi c/\omega_0$. The axes of the particle's rest system (x', ct') and of the laboratory system (x, ct) are tilted with respect to each other. The time coordinate ct as well as the space coordinate x are subdivided into units of the wave period in the laboratory system $c\tau_E = 2\pi c/\omega_E$.

The trajectory coincides with the ct'-axis

$$ct = \frac{c}{v}x, \tag{3}$$

where v is the velocity of the particle. The lines of constant phase run parallel to the x'-axis

$$ct = \frac{v}{c}x. \tag{4}$$

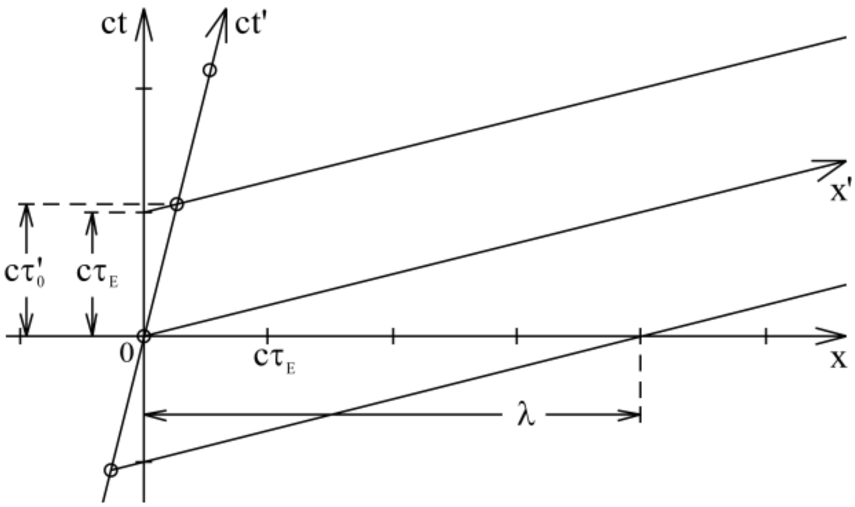

Figure 1. Minkowski diagram of the (steeply rising) trajectory of a free particle and three (slowly rising) phase lines of the escorting wave. The velocity v of the particle is arbitrarily chosen to be $v = c/4$. Details are given in the text.

Thus

$$ct = \frac{v}{c}x + c\tau_E. \tag{5}$$

and

$$ct = \frac{v}{c}x - c\tau_E. \tag{6}$$

denote the upper and lower phase lines in figure 1, respectively.

The period of the particle's internal oscillation measured in the laboratory system is given by

$$c\tau_0' = \frac{c\tau_0}{\sqrt{1-(v/c)^2}}. \tag{7}$$

On the other hand the intercept $c\tau_0'$ is the time component of the intersection of trajectory and upper phase line. Thus $c\tau_0'$ can also be expressed by

$$c\tau_0' = \frac{c\tau_E}{1-(v/c)^2}. \tag{8}$$

Combining (7), (8) and $\omega = 2\pi/\tau$ the frequency ω_E can be written in the form

$$\omega_E = \frac{\omega_0}{\sqrt{1-(v/c)^2}}. \tag{9}$$

Thus the frequency of the escort wave transforms equally like the associated mass

$$m_E = \frac{m_0}{\sqrt{1-(v/c)^2}}. \tag{10}$$

That means the relation $\hbar\omega = mc^2$ is Lorentz invariant.

The wave length of the escort wave in the laboratory system is given by the distance of the crossings of two consecutive phase lines with the x-axis. In figure 1 the middle phase line, which

Journal of Physics: Conference Series **306** (2011) 012071 doi:10.1088/1742-6596/306/1/012071

coincides with the x'-axis, crosses the x-axis at the origin. The lower phase line (6) intersects the x-axis at

$$\lambda = c\tau_E \frac{c}{v} = \frac{2\pi\hbar}{m_E v}. \tag{11}$$

The postulated wave length λ of matter waves has experimentally been verified by scattering electrons on nickel crystals [12]. It is traditionally called de Broglie wave length.

For non-relativistic movements the wave frequency $\hbar\omega_E$ may be written in the form

$$\hbar\omega_E = \frac{\hbar\omega_0}{\sqrt{1-(v/c)^2}} \approx \hbar\omega_0(1 + \frac{1}{2}\frac{v^2}{c^2}) = m_0c^2 + E_{kin}. \tag{12}$$

Thus the escort wave for a free particle

$$\psi(x,t) = \phi e^{-i(\omega_E t - kx)} \tag{13}$$

with the wave number

$$k = \frac{2\pi}{\lambda} = \frac{m_E v}{\hbar} \tag{14}$$

is a solution of the time-dependent Schrödinger equation

$$(i\hbar\frac{\partial}{\partial t} - m_0c^2)\psi(x,t) = -\frac{\hbar^2}{2m_E}\frac{\partial^2}{\partial x^2}\psi(x,t) \approx -\frac{\hbar^2}{2m_0}\frac{\partial^2}{\partial x^2}\psi(x,t). \tag{15}$$

The mass m_E may be substituted by m_0 because second order terms in v/c may be neglected if the considerations are restricted to non-relativistic problems. Contrary to the traditional notation the energy $E = \hbar\omega_E$ does not only denote the kinetic energy but also comprises the rest energy m_0c^2 of the particle. Thus the rest energy term m_0c^2, which is already known from the Dirac and Klein-Gordon equations, has to be inserted into the time-dependent Schrödinger equation. The time-independent Schrödinger equation remains unchanged because the renormalization of the wave frequency has no influence on the space dependence of the escort wave.

The escort wave (13) only depends on the energy $\hbar\omega_E$ and on the momentum $\hbar k$ of the particle, but not upon the particle's position. The variable x is an argument of the escort wave and does not denote the position of the particle at time t. Therefore a wave is in phase harmony with a particle moving on infinitely many trajectories postponed in space and time.

The escort wave concept combines Edward Nelson's stochastic mechanics [13] with de Broglie's phase wave model. Caused by the fluctuating forces the probability distribution progresses to spread in space just like wave packets in quantum mechanics.

The stochastic velocity component makes sure that the wave escorting a free particle cannot be described by a single frequency but by a (narrow) frequency band. Thus the wave can approximately be written in the form of an eigen differential

$$\Phi_{\Delta\omega}(x,t;\omega) = \frac{1}{\sqrt{\Delta\omega}}\int_{\omega+\Delta\omega/2}^{\omega-\Delta\omega/2} \phi(x,t;\omega')d\omega' \tag{16}$$

with sufficient small $\Delta\omega$. Therewith the wave packed associated with a quantum state in the continuous spectrum has a finite norm. The smearing out of the unique frequency to a frequency band also leads to a finite coherence length of the escort wave.

3. The escort wave associated with a particle moving in stationary potentials

In his doctor thesis Louis de Broglie compared the Fermat principle with the principle of Maupertius. Both principles are variational methods with the aim to find least action rays of waves and least action trajectories of particles, respectively. De Broglie came to the conclusion that the trajectories of particles and the rays of the associated waves agree. That means in the laboratory frame trajectories and phase lines have reciprocal slopes at the crossing points. Thus trajectory and phase lines locally have the same geometry as for free particles.

If no photons are emitted or absorbed the total energy $E = m_0c^2 + E_{pot} + E_{kin}$ of a particle and the frequency ω_E of the escorting wave are constant in stationary potentials. That means the vertical distance $c\tau_E$ of subsequent phase lines is equal everywhere. As for free particles the wave length $\lambda = 2\pi\hbar/(m_E v) \approx 2\pi\hbar/(m_0 v)$ only depends on the momentum $m_E v \approx m_0 v$ of the particle and not on the potential. Relations (9) and (10) can only be valid if the internal frequency ω_0 and the rest mass m_0 are replaced by potential dependent terms

$$\omega_P = \omega_0 + E_{pot}/\hbar \tag{17}$$

and

$$m_P = m_0 + E_{pot}/c^2. \tag{18}$$

That means the rest mass of a particle changes with the potential.

This change of the rest mass with the potential has also been postulated by Louis de Broglie in 1971 [14], when he carefully analyzed the phase harmony principle for particles moving in stationary potentials. Although the change of the rest mass with the potential is not explicitely discussed in textbooks the effect is well known. The mass defect in chemical and nuclear reactions is a direct consequence of the rest mass change in potentials derived from electromagnetic and strong forces, respectively. The change of the rest mass in graviational potentials has already been postulated by Albert Einstein [15] in 1915 and has often been verified in the meantime.

If relation (17) is also valid for very high energies the rest masses of two antiparticles will practically go to zero if they come very close together in case of central collisions. This effect may erraneously simulate extremely high momentum transfers indicating that for electrons and positrons the Coulomb law is still valid in the femtometer region.

If the Schrödinger equation (15) for a free particle is extended to the case of a particle moving in stationary potentials m_0c^2 has to be replaced by $m_Pc^2 = m_0c^2 + E_{pot}$ leading to the equation

$$(i\hbar\frac{\partial}{\partial t} - m_0c^2)\psi(x,t) = \Big(-\frac{\hbar^2}{2m_0}\frac{\partial^2}{\partial x^2} + E_{pot}(x)\Big)\psi(x,t) \tag{19}$$

with $\psi(x,t) = \psi(x)e^{-i\omega_E t}$.

For stationary potentials the application of the energy operator $i\hbar\partial/\partial t$ to the time dependence factor $e^{-i\omega_E t}$ leads to the usual time-independent Schrödinger equation

$$(E - m_0c^2)\psi(x) = \Big(-\frac{\hbar^2}{2m_0}\frac{\partial^2}{\partial x^2} + E_{pot}(x)\Big)\psi(x). \tag{20}$$

or if masses and energies are expressed by frequencies

$$(\omega_E - \omega_0)\psi(x) = \Big(-\frac{c^2}{2\omega_0}\frac{\partial^2}{\partial x^2} + \omega_V(x)\Big)\psi(x) \tag{21}$$

with $\omega_V(x) = E_{pot}(x)/\hbar$.

That means the escort waves associated with particles moving on classical trajectories are solutions of the Schrödinger equation.

Not only the Schrödinger equation but all formulae in quantum mechanics can be written without $\hbar$ if energies and masses are consequently expressed in form of frequencies. For example the uncertainty relation $\Delta x \Delta p \geq \hbar/2$ can be expressed in the form $\Delta x \Delta k \geq 1/2$, which is already well known from classical physics for wave packets.

Obviously Planck's constant has the character of a conversion constant comparable with the Boltzmann constant k_B. When Ludwig Boltzmann and Max Planck succeeded to express the mean energy of particles in thermal equilibrium by $k_B t$ he embedded thermodynamics into classical physics. Textbooks of electrodynamics handle the movement of particles in external fields and the radiation of moving particles but not the interaction of particles with waves radiated by the particles. The new point of view may help to understand quantum mechanics as a completion of classical physics. The mutual interaction of accelerated particles and associated waves leads to the quantization of states and to the intensity redistribution observed in diffraction experiments.

The particle-wave phenomena in microphysics arise from the interplay of classical particles and classical waves, which are coupled together via the phase harmony principle. Thus if Planck's relation $E = \hbar\omega$ is considered as an universal relation of physics quantum mechanics can be understood as an extension of classical physics.

4. Escort waves associated with bound particles

Without fluctuating forces a particle in ground state would permanently stay at rest in the potential minimum and the zero point energy would be zero. Edward Nelson [13] derived the formalism of quantum mechanics from statistical mechanics of point particles without presuming the existence of waves. After having determined the density distribution of particles he formally derived a wave. For ground states stochastic mechanics provides the probability densities and zero point energies known from quantum mechanics.

The trajectories of ground states are continuous but mainly determined by fluctuating forces. For excited states with increasing quantum numbers the stochastic velocity component becomes less and less important. The trajectories approach classical trajectories in agreement with the correspondence principle formulated by Ehrenfest [16] after having compared classical trajectories with the movement of wave packets .

Escort waves of excited states generally enclose standing waves. For three-dimensional central force potentials the wave function can be expressed by a product of three one-dimensional functions, where two of the functions are standing waves.

In the following the consequent application of the phase harmony principle to standing waves will be demonstrated on the $n = 4$ state of a linear harmonic oscillator. The vibration frequency is chosen to be $\omega_{vib} = \omega_E/20$.

Figure 2 shows phase lines of the two counter-propagating travelling waves, which constitute the standing wave, and four of the infinitely many possible classical trajectories. For all these trajectories the particle is in phase harmony with the travelling wave, whose rest system agrees with that of the particle. Because of a special kind of relativistic Doppler effect the frequency of the counter-propagating wave does not agree with the internal oscillation of the particle. The frequency shift by a factor

$$G = \frac{1 + (v/c)^2}{1 - (v/c)^2} \approx 1 + 2\Big(v/c\Big)^2 \tag{22}$$

prevents the phase coupling of the counter-propagating wave with the particle.

The trajectories in figure 2 are assumed to be deterministic. If the fluctuating velocity component is taken into account particle and wave are most often out of phase. However, driven by the phase harmony principle the particle stays close to the trajectory shown in the figure. If the deviations are too strong the particle will undergo a spontaneous transition to a lower quantum state.

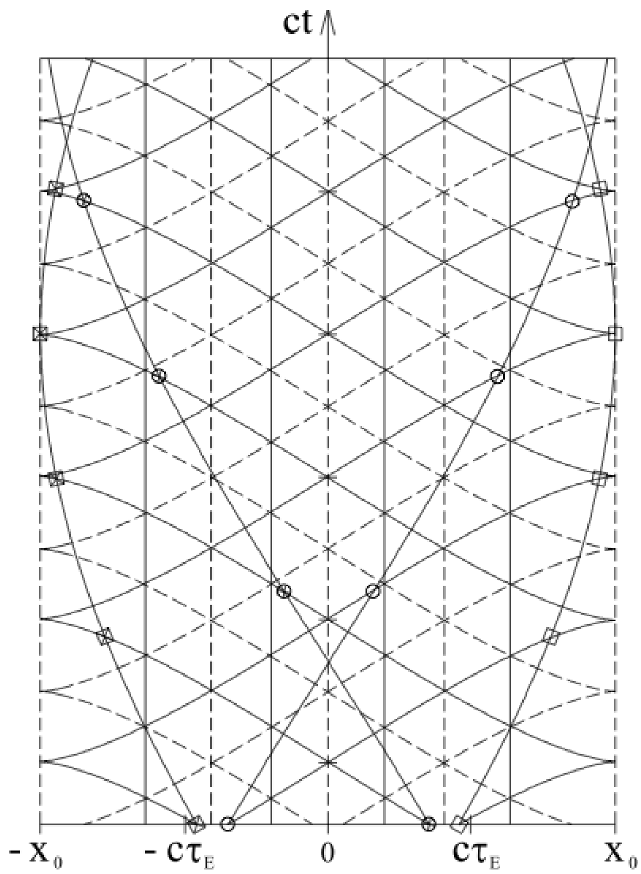

Figure 2. Four trajectories of a particle and the space-time structure of the two counter-propagating waves associated with the back and forth motion of the particle in the $n = 4$ state of a harmonic oscillator. The solid and dashed phase lines connect points with phase factors of 1 and -1, respectively. Both axes of the Minkowski-diagram are divided into units of $c\tau_E$. The distances between identical symbols indicate the particle's eigen period.

The velocities of the particle on its way back and forth are just opposite to each other. Thus their complex valued amplitudes have the same modulus at each location. Because the amplitudes are counter-rotating in the complex plane with opposite angular velocities the sum of both waves has everywhere the same phase. If the residual phase factor is arbitrarily set to one, the standing wave is real everywhere.

The intensity varies between $\psi^2(x)$ and zero, where $\psi(x)/2$ is the amplitude of the travelling waves at the location x. Without fluctuating forces the probability of finding the particle at a given location is identical with the classical probability. The fluctuations lead to a smearing out of the classical probability density, which is quite well approximated by the folding of the classical probability density with a normal distribution. The width of the normal distribution is chosen to be $\sigma = 0.48c/\sqrt{\omega_0\omega_{vib}} \approx 0.62c\tau_E$.

The occurrence of interferences is a characteristic property of waves, which has no counterpart in the probability density of particles. Whereas the standard interpretation of quantum mechanics postulates zeros in the probability density distributions the escort wave concept predicts smoothly varying probability densities.

Soon after the presentation of wave mechanics by Erwin Schrödinger Max Born [17] formulated the hypothesis, that in case of scattering experiments the wave intensity far away from

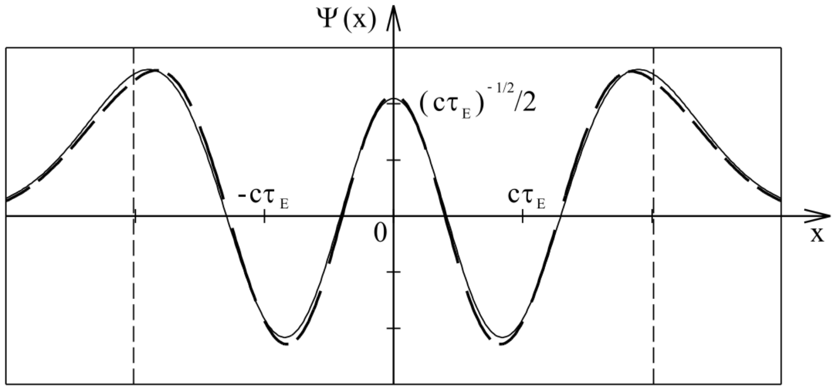

Figure 3. Comparison of the approximated space dependence of the escort wave $\phi_{approx}(x)\sin\varphi_{classic}(x)$ (dashed curve) with the Schrödinger wave function $\psi(x)$ (solid curve) for the $n = 4$ state of the harmonic oscillator. The dashed vertical lines indicate the classical turning points.

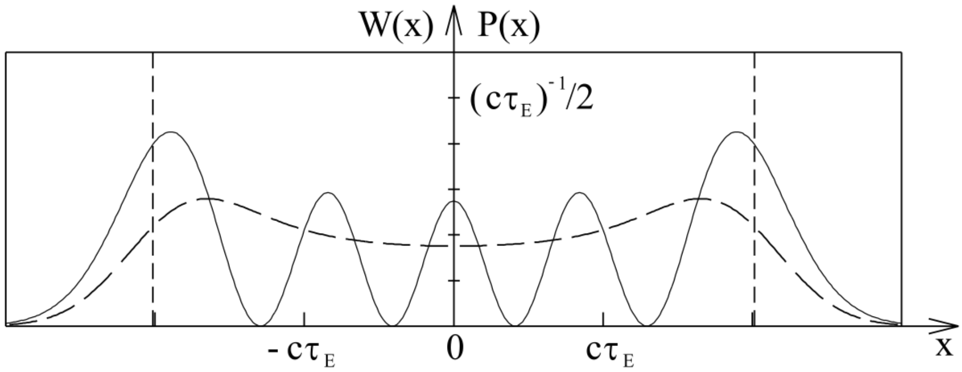

Figure 4. Comparison of the probability density of the particle derived from the approximated escort wave $P(x) = \phi^2_{approx}(x)$ (dashed curve) with the absolute square of the Schrödinger wave function $W(x) = \psi(x)\psi^*(x)$ (solid curve) for the $n = 4$ state of the harmonic oscillator. The dashed vertical lines indicate the classical turning points. Obviously the absolute square of the wave function does not give the probability of finding a particle at a given location.

the scattering zone is proportional to the probability of finding particles at a given scattering angle. This proportionality is one of the best verified laws in physics. Day by day it is confirmed by countless experiments.

Without any experimental evidence the scope of applicability of the ad-hoc assumption has been extended [18] to the probability density in bound states and in scattering zones. However, the application of the hypothesis to standing waves is not at all stringent.

A scattering experiment does not provide the probability density of a bound particle. The result is a matrix element, which contains besides the transition operator only the wave functions of the initial and final state. Thus one does not test the probability density but the amplitudes and the relative phases of the associated escort waves. Until now Born's rule cannot be verified experimentally. Recently there are even doubts on its universal validity [19]. Because Born's rule is an assumption, which has no influence on the formalism of quantum mechanics, it may be left off without any consequences. The rule must only be valid for the asymptotic behaviour of particles in order to properly describe scattering and diffraction experiments.

Experimental findings, which seem to support the probability density interpretation, are often misunderstood. So the intensity variation of the tunnel current on the surface of quantum corrals [20] can be traced back to the local variation of the exit energy. The experiments of Ketterle and his coworkers [21] showing the density modulation of particles going out from two correlated Bose-Einstein condensates can be understood as a modified double slit experiment.

Also other experiments are invoked in order to prove the probability density interpretation. In general they can be easily refused because experimental findings are misinterpreted. For example defocused high energy particle beams exhibit a characteristic density substructure. The observed beam intensity pattern comes from the diffraction of the high energy particles at the apertures arranged in the accelerator tube. In diffraction experiments one observes the intensity far from the diffraction device. In these cases the trajectories run practically parallel to the zero intensity planes. The particles, which are randomly tumbling under the influence of the fluctuating waves, are trapped in the regions of high wave intensity. This argument is no longer valid if particles just cross planes of zero intensity, which happens in the near zone of scattering processes and in excited states.

Sometimes the disturbance of the classical motion leads to a spontaneous quantum transition. But generally the particle will stay in its quantum state because the modified trajectory is in phase harmony with the original escort wave too. A single measurement process can never provide the full information about a quantum state because the particle is accidentally found in one of the infinitely many trajectories of the same class. Only by frequently repeating the measurement with the same initial conditions the structure of the wave can be fully revealed.

Experimentally only the probabilities for possible final states can be determined. The uncertainty principle prevents to find out the actual trajectory. Therefore it makes sense to subsume all trajectories belonging to the same wave to a quantum state $\psi(x)$. Quantum states of bound particles are generally found by solving eigenvalue problems in Hilbert space. In quantum mechanics many mathematically equivalent representations of the Hilbert space formalism are used.

However, the Schrödinger equation has a prominent status, because it is suited to determine the real waves escorting massive particles. Within the Schrödinger formalism the probability for a given final state can be derived from the Schrödinger wave function $\psi(x)$, which is identical with the space dependence of the escort wave. Thus the wave mechanics of Schrödinger is the appropriate method to derive quantum physics from classical physics.

That means, although the actual trajectory cannot be determined it is necessary to postulate well defined trajectories. The assumption that particle and wave coexist and are coupled to each other via the phase harmony principle reveals the proper interpretation of the Schrödinger wave function.

5. Final remarks

In the beginning of the 20th century Rutherford [22] refers in his publication to the atom model of Nagaoka [23] with electrons orbiting the nucleus. This model, which is most often attributed to Rutherford, has soon been abandoned, because in the framework of classical physics the electrons would continuously loose energy. In the (non-classical) escort wave model accelerated electrons do not loose energy, when they are in phase harmony with the associated wave. So the Rutherford model is quite adequate to descibe atomic shells, but one has to bear in mind that the orientations of the trajectories are permanently changing thus forming a rotational symmetric charge cloud around the nucleus.

The extension of the escort wave concept to many particle systems is not trivial. In case of interacting particles the ensemble of waves can no longer be described by a product of single particle waves because the movements of the particles and the associated waves are correlated. Thus the waves escorting massive particles depend on the locations and momenta of all particles

Journal of Physics: Conference Series **306** (2011) 012071 doi:10.1088/1742-6596/306/1/012071

involved. These coordinates span the configuration space.

References

[1] de Broglie L 1924 *Recherches sur la théorie des quanta* (Sorbonne Univ., Paris); de Broglie L 1925 The doctor thesis has completely been published in: *Annales de Physique* **3** 22. The doctor thesis translated into English by Kracklauer A F . Can be found on the web site of the foundation Louis de Broglie at the adress http://www.aflb.esnmp.fr/LDB-oeuvres/De_ Broglie _Kracklauer.htm.
[2] Jung K 2009 *Ann. Fond. Louis de Broglie* **34** 143
[3] Zeh D 1970 *Found. Phys.* **1** 69
[4] Zurek W H 2003 *Reviews of Modern Physics* **75** 715
[5] Bohm B 1952 *Phys. Rev.* **85** 166; Bohm B 1952 *Phys. Rev.* **85** 180
[6] de Broglie L 1956 *Une tentative d'interprétation causale et non-linéaire de la mécanique ondulatoire: la théorie de la double solution* (Gauthiers-Villars, Paris) English translation: Non-linear wave mechanics, (Elsevier, Amsterdam, 1960)
[7] de Broglie, L 1978 *Jalons pour une nouvelle microphysique* (Gauthier-Villars, Paris)154
[8] Bohm D and Vigier J P 1954 *Phys. Rev.* **96** 208
[9] Cufara-Petroni N and Vigier J P 1992 *Found. Phys.* **22** 1
[10] Vigier J P 1994 *Found. Phys.* **24**, 61
[11] Gueret P and Vigier J P 1982 *Found. Phys.* **12** 1057
[12] Davisson C and Germer L H 1927 *Phys. Rev.* **30** 705
[13] Nelson E 1966 *Phys. Rev.* **150** 1079
[14] de Broglie L 1971 *La réinterprétation de la mécanique ondulatoire* (Gauthier-Villars, Paris) 24
[15] Einstein A 1916 *Ann. Physik* **49** 769
[16] Ehrenfest P 1927 *Z. Phys.* **A 45** 455
[17] Born M 1926 *Z. Phys.* **37**, 863. English translation in Wheeler J A und Zurek W H eds., 1983 Quantum Theory and Measurement 52-55 , (Princeton University Press)
[18] Born M 1954 *Nobel lecture, Dec. 11.* Born M 1964 *in Nobel lectures Physics 1942-1962* (Elsevier Publ. Comp., Amsterdam)
[19] Khrennikov A 2009 *Found. Phys.* **39**, 997
[20] Crommie M F, Lutz C P and Eigler D M 1993 *Science* **262** 218
[21] Andrews M R, Townsend C G, Miesner H J, Durfee D S, Kurn D M and Ketterle W 1997 *Science* **275**, 637
[22] Rutherford E 1911 *Phil. Mag.* **21** 669
[23] Nagaoka H 1904 *Phil. Mag.* **7** 445